Control of Electric Motors and Drives
via Convex Optimization

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Outline

1. waveform design for electric motors
   - permanent magnet
   - induction
2. control of switched-mode converters
Waveform design for electric motors

traditionally:
- AC motors driven by sinusoidal inputs (and designed for this)\(^1\)
- based on reference frame theory, c. 1930

now:
- more computational power
- power electronics can generate near-arbitrary drive waveforms\(^2\)

our questions:
- given a motor, how to design waveforms to drive it?
- which waveform design problems are tractable? convex?

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Motor model

- $n$ windings, each with an $RL$ circuit.

- Electrical variables:
  - Voltage $v(t) \in \mathbb{R}^n$
  - Current $i(t) \in \mathbb{R}^n$
  - Flux $\lambda(t) \in \mathbb{R}^n$
the rotor has

- torque $\tau(t)$
- speed $\omega = \text{const.}$ (high inertia mech. load)
- position $\theta(t) = \omega t$

goal is to manipulate $\nu$ to control $\tau$
Stored energy

- stored magnetic energy is $E(\lambda, \theta)$
  - magnetic coupling depends on mechanical position
- $E$ is $2\pi$-periodic in $\theta$
- inductance equation relates current and flux:
  \[
  i = \nabla_\lambda E(\lambda, \theta)
  \]
- torque given by
  \[
  \tau = -\frac{\partial}{\partial \theta} E(\lambda, \theta)
  \]
- in general, both are nonlinear in $\lambda$
Torque

- the average torque is:

\[ \bar{\tau} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \tau(t) \, dt \]

- torque ripple is

\[ r = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} (\tau(t) - \bar{\tau})^2 \, dt \]
$R \in S_++^n$ is the (diagonal) resistance matrix

- resistive power loss is $i^T R i$
- average power loss is

$$\rho_{loss} = \lim_{T \to \infty} \frac{1}{T} \int_0^T i^T R i \, dt$$
Circuit dynamics

- dynamics from Kirchoff’s voltage law, Faraday’s law:

\[ v(t) = Ri(t) + \dot{\lambda}(t) \]

- dynamics coupled across windings by inductance equation

\[ i = \nabla_\lambda E(\lambda, \theta). \]
often, winding voltages $v$ not controlled directly
(e.g., wye/delta windings, windings contained in rotor)
indirect control through terminal voltages $u(t) \in \mathbb{R}^m$

$$Ci(t) = 0, \quad v(t) = C^T e(t) + Bu(t),$$

$C \in \mathbb{R}^{p \times n}$ is the connection topology matrix
$B \in \mathbb{R}^{n \times m}$ is the voltage input matrix
$e(t) \in \mathbb{R}^p$ are floating node voltages
Winding connection examples

\[ C_i(t) = 0, \quad v(t) = C^T e(t) + Bu(t), \]

- simple delta, wye, and independent winding connections
- some windings may be controlled only through induction
  - e.g., windings on the rotor
Optimal waveform design

- waveform design problem:

  minimize \( p_{\text{loss}} + \gamma r \)

  subject to \( \bar{\tau} = \tau_{\text{des}}, \)

  torque equation

  inductance equation

  circuit dynamics

  winding pattern

- variables are \( i, v, u, e, \lambda, \tau \) (all functions on \( \mathbb{R}_+ \))

- problem data:
  - tradeoff parameter \( \gamma \geq 0 \)
  - resistance matrix \( R \in S^{++}_n \)
  - energy function \( E : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \)
  - shaft speed \( \omega \in \mathbb{R} \)
  - desired torque \( \tau_{\text{des}} \in \mathbb{R} \)
  - winding connection parameters \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{p \times n} \)
nonconvex in general, due to nonlinear torque and inductance equations

- problem data $2\pi$-periodic, but periodicity of solution not known
  - in practice, solutions often *not* $2\pi$-periodic in $\theta$
Permanent magnet motor

- magnets in rotor change magnetic flux through windings as they pass, producing voltage across the windings
- by simultaneously pushing current through the windings, electrical energy is extracted (or injected)
Permanent magnet motor

- energy function is quadratic:

\[ E(\lambda, \theta) = \lambda^T A \lambda + b(\theta)^T \lambda \]

(quadratic part independent of rotor angle)

- inductance equation is linear:

\[ \lambda = L i + \lambda_{\text{mag}}(\theta) \]

\( L \) is the inductance matrix, \( \lambda_{\text{mag}} \) is the flux due to rotor magnets

- torque equation is affine:

\[ \tau = k(\theta)^T i + \tau_{\text{cog}}(\theta) \]

\( k(\theta) \) is the motor constant, \( \tau_{\text{cog}} \) is the cogging torque
Permanent magnet motor

- dynamics, with $\lambda$, are

$$v(t) = Ri(t) + \dot{\lambda}(t)$$

- eliminating $\lambda$:

$$v(t) = Ri(t) + L \frac{di}{dt}(t) + \omega k(\theta)$$
Permanent magnet motor, waveform design

- optimal waveform design problem is convex
- $2\pi$-periodicity of problem data with convexity implies $2\pi$-periodicity of a solution, if one exists.\(^3\)

\(^3\)Boyd, Vandenberghe. *Convex Optimization*, page 189. 2004
Permanent magnet motor, waveform design

- waveform design problem:

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2\pi} \int_{0}^{2\pi} i(\theta)^T R i(\theta) \, d\theta + \gamma \frac{1}{2\pi} \int_{0}^{2\pi} (\tau(\theta) - \tau_{\text{des}})^2 \, d\theta \\
\text{subject to} & \quad \frac{1}{2\pi} \int_{0}^{T} \tau(\theta) \, d\theta = \tau_{\text{des}} \\
& \quad \tau = k(\theta)^T i + \tau_{\text{cog}}(\theta) \\
& \quad v(\theta) = Ri(\theta) + \omega Li'(\theta) + \omega k(\theta) \\
& \quad Ci(\theta) = 0 \\
& \quad v(\theta) = C^T e(\theta) + Bu(\theta)
\end{align*}
\]

- variables are \(i, v, u, e, \tau\) (all functions on \([0,2\pi]\))
Permanent magnet motor, waveform design

- a periodic linear-quadratic control problem
  - can discretize, solve by least squares
- in fact, many extensions retain convexity:
  - voltage limits \(|u(\theta)| \leq u_{\text{max}}\)
  - current limits \(|i(\theta)| \leq i_{\text{max}}\)
  - nonquadratic definitions of torque ripple
- extensions typically involve solving a quadratic program
- more discussion in paper\(^4\):
  - extensions/variations
  - custom fast solver → online waveform generation

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Example

- $\gamma = 2 \text{ W/Nm}^2$
- Left: $\omega = 300 \text{ rad/s}$, right: $\omega = 400 \text{ rad/s}$
Induction motor

- rotor magnets replaced by more windings, which act as electromagnets (with current)
- rotor current produced by magnetic induction (using stator currents)
Induction motor

- Energy function is again quadratic:

\[ E(\lambda, \theta) = \lambda^T A(\theta) \lambda \]

quadratic part dependent on \( \theta \) (affine part omitted for simplicity)

- Inductance equation is linear:

\[ \lambda = L(\theta)i \]

- Torque is (indefinite) quadratic:

\[ \tau = -i^T L'(\theta)i \]
Induction motor, maximum torque problem

- general waveform design problem intractable
- we focus on the maximum torque problem ($\gamma = 0$):
  - torque ripple penalty disappears
  - maximize average torque (a nonconvex quadratic function)
  - power loss constraint (a convex quadratic function)
Induction motor, maximum torque problem

- waveform design problem:

\[
\begin{align*}
\text{maximize} & \quad \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} -i(t)^T L'(\omega t) i(t) \, dt \\
\text{subject to} & \quad \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} i(t)^T R i(t) \, dt \leq p_{\text{loss}} \\
& \quad v(t) = R i(t) + \dot{\lambda}(t) \\
& \quad C i(t) = 0 \\
& \quad v(t) = C^T e(t) + B u(t) \\
& \quad \lambda(t) = L(\omega t) i(t)
\end{align*}
\]

- variables are \( i, v, u, e, \lambda \) (all functions on \( \mathbb{R}_+ \))

- equivalent to minimizing \( p_{\text{loss}} \) with average torque constraint
Induction motor, maximum torque problem

- can be converted to a nonconvex linear-quadratic control problem with a quadratic constraint
  - strong duality holds
  - original proof due to Yakubovich\textsuperscript{5}
- further details in our paper\textsuperscript{6}
  - equivalent semidefinite program (SDP)
  - method for constructing optimal waveforms from SDP solution
  - proof of tightness

\textsuperscript{6}Moehle, Boyd. \textit{Maximum Torque-per-Current Control of Induction Motors via Semidefinite Programming}. 2016.
Example

traditional, sinusoidally wound, 5-phase motor with wye winding:

desired torque $\tau_{\text{des}} = 5 \text{ Nm}$, speed $\omega = 50 \text{ rad/s}$
Example

power loss is 11 W per Nm torque produced
Stator fault

Same motor, with open-phase fault:
Stator fault

Power loss is 14 W per Nm torque produced
Outline

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   - induction

2. control of switched-mode converters
Controlling switched-mode converters

input are switch configurations

traditionally:\textsuperscript{7}

1. make discrete input continuous, by considering averaged switch on-time (‘duty cycle’)
2. choose a duty cycle corresponding to desired equilibrium
3. linearize the resulting system around equilibrium, use linear control

now:

state $x_t \in \mathbb{R}^n$ contains inductor currents, capacitor voltages
   – can be augmented to contain, e.g., reference signal
for each switch configuration, we have a linear circuit
switched-affine dynamics:

$$x_{t+1} = A^{u_t} x_t + b^{u_t}, \quad t = 0, 1, \ldots,$$

dynamics specified by $A^i, b^i$ in mode $i$
control input is the mode $u_t \in \{1, \ldots, K\}$
may include mode restrictions (e.g., for a diode)
Switched-affine control

- switched-affine control problem is

\[
\begin{align*}
\text{minimize} & \quad \sum_{t=1}^{T} g(x_t) \\
\text{subject to} & \quad x_{t+1} = A^{u_t} x_t + b^{u_t} \\
& \quad x_0 = x_{\text{init}} \\
& \quad u_t \in \{1, \ldots, K\}
\end{align*}
\]

- constraints hold for all \( t \)
- variables are \( u_t \) and \( x_t \in \mathbb{R}^n \)
- problem data are dynamics \( A^i, b^i \), function \( g \), and initial condition \( x_{\text{init}} \)
- can be solved by trying out \( K^T \) trajectories
‘Solution’ via dynamic programming

- Bellman recursion: find functions $V_t$ such that

$$V_t(x) = \min_{u \in \{1, \ldots, K\}} g(x) + V_{t+1}(A^u x + b^u)$$

for all $x$, for $t = T - 1, \ldots, 0$

- final value function $V_T = g$

- optimal problem value is $V_0(x_{\text{init}})$ at initial state $x_{\text{init}}$

- in general, intractable to compute (or store) $V_t$
Model predictive control

- idea: solve switched-affine control problem, implement first control action $u_0$, measure new system state, and repeat
- called *model predictive control* (MPC) or *receding horizon control*
- given $V = V_1$, MPC policy satisfies

$$
\phi_{mpc}(x) \in \arg\min_{u \in \{1, \ldots, K\}} V(A^u x + b^u)
$$

(ties broken arbitrarily)
Approximate dynamic programming policy

- In practice, MPC policy only works for $T$ small
- (system response time measured in $\mu$s)
- Instead, approximate $V$ as a quadratic function $\hat{V}$
- Given $\hat{V}$, ADP policy satisfies

$$\phi_{adp}(x) \in \arg\min_{u \in \{1, \ldots, K\}} \hat{V}(A^u x + b^u)$$

- Evaluating $\phi_{adp}$ requires evaluating a few quadratic functions
How to obtain $\hat{V}$?

- quadratic lower bounds on $V$ can be found via semidefinite programming\(^8\)
- compute $V(x^{(i)})$ for many states $x^{(i)}$, fit best quadratic function $\hat{V}$
  - we used this method
  - subproblems solved using methods described in paper\(^9\)
- use exact value function for approximate linear control problem (e.g., linear-quadratic control)
  - provides a link to traditional methods

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state $x_t$ are inductor currents and capacitor voltages, and desired output current phasors

- cost function is deviation of output currents from desired (sinusoidally-varying) values

- model parameters $V_{dc} = 700$ V, $L_1 = 6.5 \mu$H, $L_2 = 1.5 \mu$H, $C = 15 \mu$F, $V_{load} = 300$ V, and desired output current amplitude $I_{des} = 10$ A.

- sampling time 30 $\mu$s
### Result

<table>
<thead>
<tr>
<th>Policy</th>
<th>State cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADP policy</td>
<td>0.70</td>
</tr>
<tr>
<td>MPC policy, $T = 1$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>MPC policy, $T = 2$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>MPC policy, $T = 3$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>MPC policy, $T = 4$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>MPC policy, $T = 5$</td>
<td>0.45</td>
</tr>
</tbody>
</table>

- for $T < 5$ MPC policy is unstable
- running MPC with $T = 5$ takes several seconds on PC
- ADP takes few hundred flops (can be carried out in $\mu$s)
Result

In steady state:

![Graph showing input current, output current, and capacitor voltage over time.](image-url)
Conclusions

- unconventional motors (asymmetrical, nonsinusoidally-wound, non-rotary) can be controlled using optimization, by designing the waveform to the motor
- modern techniques can be used to generate optimal controllers for power electronic converters, which
  - have fast response
  - can easily incorporate constraints
  - are intuitive to understand and tune
  - make good use of modern microprocessor capabilities
Sources for thesis

- **motors**

- **converters**
Other work

- published:

- unpublished:
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