Linearly Constrained Separable Optimization using PiecewiseQuadratics.jl and LCSO.jl

Nicholas Moehle **Ellis Brown** Mykel Kochenderfer

BlackRock AI Labs

JuliaCon JuMP-dev Workshop

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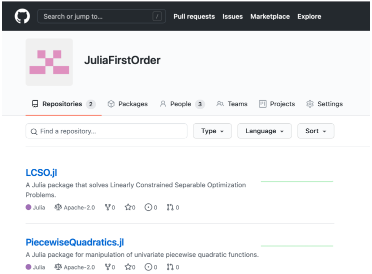
Outline

PiecewiseQuadratics.jl

LCSO.jl

JuliaFirstOrder

Portfolio Optimization



Portfolio Construction as
Linearly Constrained Separable Optimization

Nicholas Moehle Jack Gindi Stephen Boyd
Mykel J. Kochenderfer

March 10, 2021

[2103.05455]

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  - derivative
  - convex envelope
  - proximal operator
  - ...

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- Implements several methods useful for optimization
  - sum
  - derivative
  - convex envelope
  - proximal operator
  - ...
- Applications to cost functions
  - Higher fidelity
  - Computationally tractable

Example

$$f(x) = \begin{cases} x^2 - 3x + 3 & \text{if } x \in [-\infty, 3] \\ -x + 3 & \text{if } x \in [3, 4] \\ 2x^2 - 20x + 47 & \text{if } x \in [4, 6] \\ x - 7 & \text{if } x \in [6, 7.5] \\ 4x - 29 & \text{if } x \in [7.5, \infty] \end{cases}$$

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```julia
using PiecewiseQuadratics
f = PiecewiseQuadratic([
  # BoundedQuadratic(  lb,  ub,   p,     q,     r),
    BoundedQuadratic(-Inf, 3.0, 1.0,  -3.0,   3.0),
    BoundedQuadratic( 3.0, 4.0, 0.0,  -1.0,   3.0),
    BoundedQuadratic( 4.0, 6.0, 2.0, -20.0,  47.0),
    BoundedQuadratic( 6.0, 7.5, 0.0,   1.0,  -7.0),
    BoundedQuadratic( 7.5, Inf, 0.0,   4.0, -29.0)
]);
```

Plot

```
using Plots
plot(get_plot(f); ...)
plot!(get_plot(simplify(envelope(f))); ...)
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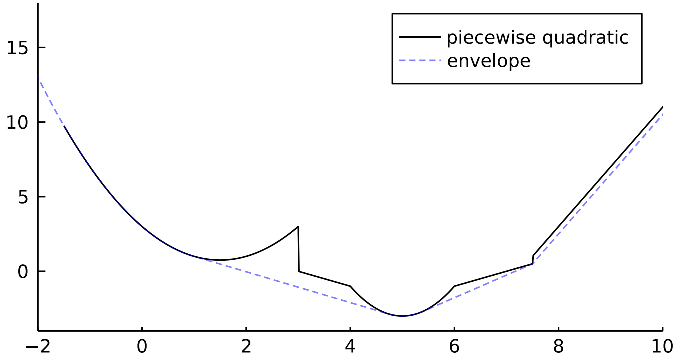

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Linearly Constrained Separable Optimiaziton

A *linearly constrained separable optimization* (LCSO) problem:

$$\begin{array}{ll} \text{minimize} & f(x) = \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & Ax = b, \end{array} \tag{1}$$

where

- the decision variable is $x \in \mathbf{R}^n$
- the parameters are $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, and the functions f_i

[1] See: arXiv:2103.05455

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LCSO problems can be solved using ADMM.[1]

- Exactly, if the f_i are convex
- Approximately, if they aren't

[1]See: arXiv:2103.05455

Extended-Form LCSO Problems

An extended-form LCSO problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & Ax = b, \end{array} \tag{2}$$

where

- $q \in \mathbf{R^n}$, $P \in \mathbf{S^n_+}$ is a symmetric positive semidefinite matrix
- f_i are piecewise quadratic functions

[2] See: arXiv:2103.05455

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- $q \in \mathbf{R^n}$, $P \in \mathbf{S^n_+}$ is a symmetric positive semidefinite matrix
- f_i are piecewise quadratic functions
- Extended-form LCSO problems can be reduced to standard LCSO problem (using an eigendecomposition)[2]

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- In practice, converges to a moderately accurate solution quickly (even if the f_i are very complicated)
- The separable functions f_i must be piecewise quadratic (see `PiecewiseQuadratics.jl`)
- (In theory, could be extended to any f_i that support a `prox` method)

Example

```
using LCSO
using PiecewiseQuadratics

n = 4  # num features
m = 2  # num constraints
```

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X = rand(n, n)
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P = X'X
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Recall the extended-form (2)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Px + q^T x + \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & Ax = b. \end{array}$$

Example

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# x_1 has to be ∈ [−1, 2] ∪ [2.5, 3.5]
#   with quadratic penalty ∈ [−1, 2]
#   and linear penalty ∈ [2.5, 3.5]
f1 = PiecewiseQuadratic([
  BoundedQuadratic( -1,  2,1,0,0),
  BoundedQuadratic(2.5,3.5,0,1,0)
])
```

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params = AdmmParams(P, q, A, b, f)
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# solve
vars, stats = optimize(params)
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print(vars.x)  # optimal x
# [-0.0493, 1.218, -1.932, 1.231]
```

.

Applications

- Portfolio optimization

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- Radiation treatment planning

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Outline

JuliaFirstOrder

- GitHub organization for first-order methods in Julia
  - https://github.com/JuliaFirstOrder
- Plan to migrate several related packages
  - ProximalOperators.jl
  - ProximalAlgorithms.jl
  - StructuredOptimization.jl
  - ...
- Thank you to Miles Lubin for the introductions!

Outline

Portfolio Optimization

The original mean–variance portfolio optimization problem of Markowitz, *i.e.*, "Risk Minimization"

$$\begin{array}{ll} \text{minimize} & \gamma x^T \Sigma x - \mu^T x \\ \text{subject to} & \mathbf{1}^T x = 1, \end{array} \qquad (3)$$

where

- $x \in \mathbf{R}^n$ is the fraction of the portfolio value in each of n assets

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- $\mu \in \mathbf{R}^n$ is the expected return forecast for the n assets, meaning $\mu^T x$ is the expected portfolio return

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- $\Sigma \in \mathbf{S}^n_{++}$ is the asset return covariance matrix, meaning $x^T \Sigma x$ is the variance of the portfolio return
- $\gamma > 0$ is the risk aversion parameter

...With Separable Costs!

The portfolio optimization problem with separable costs:

$$\begin{array}{ll} \text{minimize} & \gamma x^T \Sigma x - \mu^T x + \sum_{i=1}^n f_i(x_i) \\ \text{subject to} & \mathbf{1}^T x = 1, \end{array} \tag{4}$$

where

- f_i are asset-level penalties, like taxes and trading costs, and are piecewise quadratic

Separable cost examples

- Trading costs:

$$f_i^{\mathtt{trd}}(x_i) = s_i|x_i - x_{\mathrm{init},i}| + \begin{cases} 0 & x_i = x_{\mathrm{init},i} \\ c_i^{\mathtt{trd}} & \text{otherwise} \end{cases}$$

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$$f_i^{\mathrm{hold}}(x_i) = \begin{cases} 0 & x_i = 0 \\ c_i^{\mathrm{hold}} & \text{otherwise} \end{cases}$$

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- Position limits:

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- Combinations of these:

$$f_i(x_i) = f_i^{\mathrm{trd}}(x_i) + f_i^{\mathrm{hold}}(x_i) + f_i^{\mathrm{lim}}(x_i)$$

Relaxation

- Problems are often nonconvex
- One way to handle this: **Relax** the problem by replacing f_i with its **convex envelope**

$$f_i^{**}(x) = \sup\{g(x) \mid g \text{ is convex and } g(x) \leq f_i(x),\ x \in \mathbf{dom}(f_i)\},$$

then use the result to recover a solution to the original problem

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  then use the result to recover a solution to the original problem
- Easy to compute when f_i are piecewise quadratic
- Supported by `PiecewiseQuadratics.jl`

Questions?