Embedded Convex Optimization with CVXPY

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Outline

Optimization

optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x;\theta) \\ \text{subject to} & f_i(x;\theta) \leq 0, \quad i = 1, \ldots, m \end{array}$$

- decision variable x (a vector)
- objective function f_0
- constraint functions f_i, for $i = 1, \ldots, m$
- parameter(s) θ

the solution $x^\star$ minimizes the objective over all vectors satisfying the constraints.

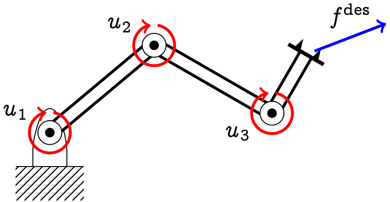
Convex optimization

- all f_i are convex (have nonnegative curvature)
- includes least-squares, linear/quadratic programming, ...
- can obtain global solution quickly and reliably
- mature software available (open source and commercial)

Domain-specific languages for convex optimization

- used to specify (and solve) convex problems
- problem constructed out of:
  - variables
  - constants
  - parameters (value fixed at solve time, may change between solves)
  - functions from a library
- specified problem mapped to a solver format (*e.g.*, conic form)
- makes prototyping easier
- DSLs include CVXPY, CVX, Convex.jl, YALMIP, ...

Example: actuator allocation



- $u \in \mathbf{R}^m$ are actuator values
- generate force $f^{\mathrm{des}} \in \mathbf{R}^n$ according to $f^{\mathrm{des}} = Au$
- A depends on system configuration (*e.g.*, joint angles)
- want u small, near previous value u^{prev}
- actuator limits $u^{\mathrm{min}} \leq u \leq u^{\mathrm{max}}$

Actuator allocation problem

$$
\begin{array}{ll}
\text{minimize} & \|u\|_1 + \lambda\|u - u^{\mathrm{prev}}\|_2^2 \\
\text{subject to} & Au = f^{\mathrm{des}} \\
& u^{\min} \leq u \leq u^{\max}
\end{array}
$$

- variable is u
- constants are $u^{\min}$, $u^{\max}$, $\lambda > 0$
- parameters are A, f^{des}, u^{prev}

Actuator allocation in CVXPY

CVXPY code:

```
u = Variable(10)
A = Parameter((6, 10), value=A_val)
f_des = Parameter(6, value=f_val)
u_prev = Parameter(10, value=u_val)

prob = Problem(Minimize(norm(u, 1)
                    + lambda*sum_square(u - u_des),
               [A*u == f_des, u_min <= u, u <= u_max])
prob.solve()
```

Under the hood: canonicalization

CVXPY transforms original problem

$$
\begin{array}{ll}
\text{minimize} & \|u\|_1 + \lambda\|u - u^{\mathrm{prev}}\|_2^2 \\
\text{subject to} & Au = f^{\mathrm{des}} \\
& u^{\mathrm{min}} \leq u \leq u^{\mathrm{max}}
\end{array}
$$

into equivalent standard-form QP:

$$
\begin{array}{ll}
\text{minimize} & \begin{bmatrix} u \\ t \end{bmatrix}^T \begin{bmatrix} \lambda I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix} + \begin{bmatrix} -2\lambda u^{\mathrm{prev}} \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} u \\ t \end{bmatrix} \\
\text{subject to} & \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix} = f^{\mathrm{des}} \\
& \begin{bmatrix} I & 0 \\ -I & 0 \\ I & -I \\ -I & -I \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix} \leq \begin{bmatrix} u^{\mathrm{max}} \\ -u^{\mathrm{min}} \\ 0 \\ 0 \end{bmatrix}
\end{array}
$$

with variable $(u, t) \in \mathbf{R}^{2m}$

Outline

Convex optimization

Embedded convex optimization

DSLs for embedded convex optimization

Examples

Embedded convex optimization

- solve same problem many times, using different parameter values
- real-time deadlines (milliseconds, microseconds)
- small software footprint
- extreme reliability
- no babysitting

Embedded optimization applications

- automatic control
  - actuator allocation
  - model predictive control
  - trajectory generation
- signal processing
  - moving-horizon estimation
- energy
  - battery management
  - hybrid vehicle control
  - HVAC control
- finance
  - quantitative trading

Embedded convex optimization solvers

- a solver maps parameters to solution (for a specific problem family)
- some (open-source) examples:
  - ECOS (2013)
  - qpOASES (2014)
  - OSQP (2016)
- typically written in C or C++
- special attention to memory allocation, division, ...
- solve one problem repeatedly with different parameters
  - symbolic step, followed by numerical step (ECOS, OSQP)
  - factorization caching (OSQP)

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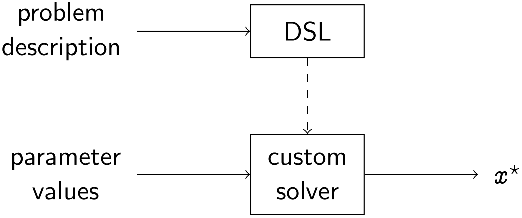
DSLs for embedded convex optimization

- parse a parametrized problem, generate a custom solver



- can re-solve problem with new parameters
- problem structure fixed (including parameter size)
- solver code optimized during code generation

DSLs for embedded convex optimization

what can we pre-compute?

- reduction to standard form (canonicalization)
- sparsity patterns of problem data
- efficient permutation for sparse matrix factorization
- factorization fill-in
- in some cases, can cache matrix factorizations

for small–medium problems, saved overhead is (very) significant

CVXGEN

- Mattingley, Boyd (2012)
- code generation for quadratic programs
- generates library-free C source
- built-in backend solver (interior point method)
- explicit coding style
  - very fast for small problems
  - code size scales poorly past a few thousand scalar parameters
- used in industry, *e.g.*, SpaceX

CVXPY-codegen

- Moehle, Boyd
- Python-based (an extension of CVXPY)
- generates library-free, embedded C source
- interchangeable backend solvers:
  - ECOS (interior point)
  - OSQP (ADMM), soon
- code size / runtime scale gracefully with problem description size
  - (but slower than CVXGEN for very small problems)
- open source
- makes Python interface for generated solver

Canonicalization

parametrized problem

$$\begin{array}{ll} \text{minimize} & f_0(x;\theta) \\ \text{subject to} & f_i(x;\theta) \leq 0, \quad i = 1, \ldots, m \end{array}$$

converted to a QP:

$$\begin{array}{ll} \text{minimize} & z^T P(\theta) z + q(\theta)^T z \\ \text{subject to} & A(\theta) z + b(\theta) \geq 0 \end{array}$$

- z is (augmented) decision variable
- P, q, A, and b depend on parameters
- solution $x^\star$ recovered from $z^\star$
- canonicalization step during code generation
- (conversion to conic problems is similar)

Storing P, q, A, and b in CVXGEN

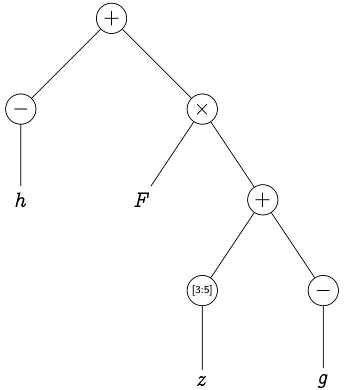
- $P(\theta)$, $q(\theta)$, $A(\theta)$, and $b(\theta)$ must be updated before each solve
- CVXGEN uses an explicit style:

```
b[13] = -(G[300]*h[0] + G[301]*h[1] + G[302]*h[2] + G[303]*h[3]
        + G[304]*h[4] + G[305]*h[5] + G[306]*h[6] + G[307]*h[7]
        + G[308]*h[8] + G[309]*h[9] + G[310]*h[10] + G[311]*h[11]
        + G[312]*h[12] + G[313]*h[13] + G[314]*h[14] + G[315]*h[15]
        + G[316]*h[16] + G[317]*h[17] + G[318]*h[18] + G[319]*h[19]
        + G[320]*h[20] + G[321]*h[21] + G[322]*h[22] + G[323]*h[23]
        + G[324]*h[24] + G[325]*h[25] + G[326]*h[26] + G[327]*h[27]
        + G[328]*h[28] + G[329]*h[29] + G[330]*h[30] + G[331]*h[31]
        + G[347]*h[47] + G[348]*h[48] + G[349]*h[49]);
```

- fast, but limits CVXGEN to small problems

Affine parse tree

- or, store $A(\theta)z + b(\theta)$ as an affine parse tree:



  for $F(z_{[3:5]} - g) - h$
- $P(\theta)$ and $q(\theta)$ represented similarly

Affine parse tree in CVXPY-codegen

- each subtree is an affine function
  - recursively walk the tree to build $A(\theta)$ and $b(\theta)$
  - at each node, carry out operation directly on subtree coefficients
- recursion is unwrapped in generated code:

```
neg(param_h, node1_offset);
index_coeff(var_z, node1_var_z);
neg(param_g, node2_offset);
matmul(param_F, node1_var_z, node3_var_z);
matmul(param_F, node2_offset, node3_offset);
sum(node1_offset, node3_offset);
```

- sparsity patterns fixed during code generation
- better scalability than explicit methods; still fast for small problems

C code structure

structure of generated code:

1. `init()`: initializes backend solver, allocate solver memory (if needed)
2. `solve()`: takes in parameters, solves problem
3. `cleanup()`: `frees` solver memory

practical usage: `init` once, then `solve` many times in a loop

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Actuator allocation: code generation

$$\begin{array}{ll} \text{minimize} & \|u\|_1 + \lambda \|u - u^{\text{prev}}\|_2^2 \\ \text{subject to} & Au = f^{\text{des}} \\ & u^{\text{min}} \leq u \leq u^{\text{max}} \end{array}$$

Python code:

```
u = Variable(10, name='u')
A = Parameter((6, 10), name='A')
f_des = Parameter(6, name='f_des')
u_prev = Parameter(10, name='u_prev')

prob = Problem(Minimize(norm(u, 1)
                  + lambda*sum_square(u - u_des),
               [A*u == f_des, u_min <= u, u <= u_max])
codegen(prob, 'target_directory')
```

Actuator allocation: generated code

```
typedef struct params_struct{
    double A[6][10];
    double f_des[6];
    double u_prev[10];
} Params;

typedef struct vars_struct{
    double u[10];
} Vars;

typedef struct work_struct{
      ...
} Work;

void cg_init(Work *work);
int cg_solve(Params *params, Work *work, Vars *vars);
void cg_cleanup(Work *work);
```

Actuator allocation: usage example

usage example:

```
int main(){
    Params params;
    Vars vars;
    Work work;
    cg_init(&work); // Initialize solver.

    while(1){
        update_params(&params); // Get new data, update parameters.
        cg_solve(&params, &work, &vars); // Solve problem.
        implement_vars(&vars); // Implement the solution.
    }
}
```

Actuator allocation: results

CVXPY-codegen (with ECOS):

- solve time: 200 μs (ECOS is $>$ 95 % of this)
- memory usage: 23 kB
- code size: 80 kB

Model predictive control

- control the linear dynamical system

$$x_{t+1} = Ax_t + Bu_t$$

  over T time periods
- input constraints $\|u_t\|_\infty \leq u^{\max}$
- problem is:

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{T-1} \|x_t\|_2^2 + \|u_t\|_2^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 0, \ldots, T-1 \\ & \|u_t\|_\infty \leq u^{\max}, \quad t = 0, \ldots, T-1 \\ & x_0 = x_{\text{init}} \\ & x_T = 0 \end{array}$$

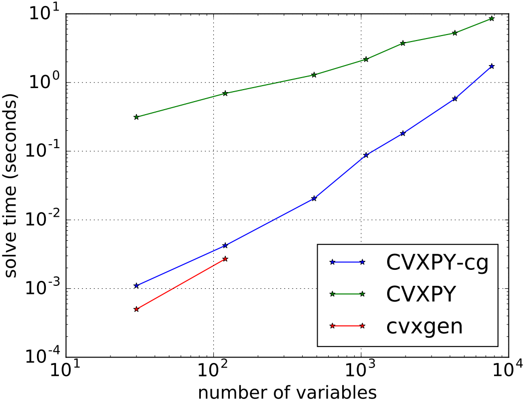
Model predictive control: Python code

```
A = Parameter((n, n), name='A')
B = Parameter((n, m), name='B')
x0 = Parameter(n, name='x0')
u_max = Parameter(name='u_max')
x = Variable((n, T+1), name='x')
u = Variable((m, T), name='u')

obj = 0
constr = [x[:,0] == x0, x[:,-1] == 0]
for t in range(T):
    constr += [x[:,t+1] == A*x[:,t] + B*u[:,t],
               norm(u[:,t], 'inf') <= u_max]
    obj += sum_squares(x[:,t]) + sum_squares(u[:,t])

prob = Problem(Minimize(obj), constr)
codegen(prob, 'target_directory')
```

Model predictive control: solve times



- inputs, states, and horizon in ratio $1:2:5$
- backend solver for CVXPY-codegen was ECOS

Other problems

Solve times for other problems (in milliseconds)

problem	CVXGEN	CVXPY-cg	CVXPY
`battery1`	.303	1.46	509
`battery2`	1.52	4.50	3112
`battery3`	—	61.5	55591
`portfolio1`	0.342	1.099	61.6
`portfolio2`	1.127	2.684	62.9
`portfolio3`	—	79.8	152.9
`lasso1`	0.136	1.40	44.4
`lasso2`	—	10.51	73.3
`lasso3`	—	52.37	185.4

Conclusion

- DSLs make (embedded) convex optimization easy to use
- convex optimization for real-time applications
- automatic code generation makes deployment easy