Control of Electric Motors and Drives via Convex Optimization

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Outline

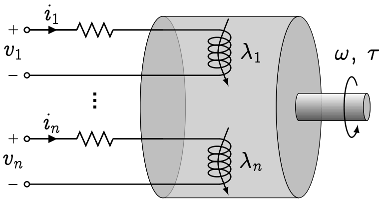
1. waveform design for electric motors
   - permanent magnet
   - induction
2. control of switched-mode converters

Waveform design for electric motors

- traditionally:
  - AC motors driven by sinusoidal inputs (and designed for this)[1]
  - based on reference frame theory, c. 1930
- now:
  - more computational power
  - power electronics can generate near-arbitrary drive waveforms[2]
- our questions:
  - given a motor, how to design waveforms to drive it?
  - which waveform design problems are tractable? convex?

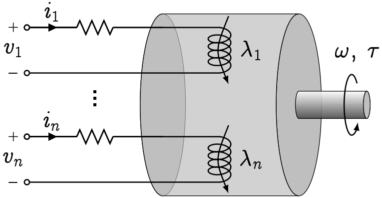
[1] Hendershot, Miller. *Design of Brushless Permanent-Magnet Machines*. 1994.
[2] Wildi. *Electrical Machines, Drives and Power Systems*. 2006.

Motor model



- n windings, each with an RL circuit.
- electrical variables:
  - voltage $v(t) \in \mathbf{R}^n$
  - current $i(t) \in \mathbf{R}^n$
  - flux $\lambda(t) \in \mathbf{R}^n$

Motor model



- the rotor has
  - torque $\tau(t)$
  - speed $\omega =$ const. (high inertia mech. load)
  - position $\theta(t) = \omega t$
- goal is to manipulate v to control τ

Stored energy

- stored magnetic energy is $E(\lambda, \theta)$
  - magnetic coupling depends on mechanical position
- E is 2π-periodic in θ
- inductance equation relates current and flux:

$$i = \nabla_\lambda E(\lambda, \theta)$$

- torque given by

$$\tau = -\frac{\partial}{\partial \theta} E(\lambda, \theta)$$

- in general, both are nonlinear in λ

Torque

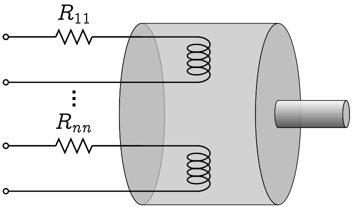
- the average torque is:

$$\bar{\tau} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \tau(t)\ dt$$

- torque ripple is

$$r = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\tau(t) - \bar{\tau}\right)^2\ dt$$

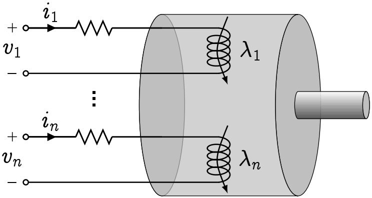
Power loss



- $R \in \mathbf{S}^n_{++}$ is the (diagonal) resistance matrix
- resistive power loss is $i^T R i$
- average power loss is

$$p_{\text{loss}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T i^T R i \, dt$$

Circuit dynamics

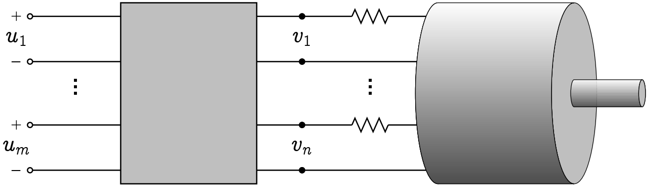


- dynamics from Kirchoff's voltage law, Faraday's law:

$$v(t) = Ri(t) + \dot{\lambda}(t)$$

- dynamics coupled across windings by inductance equation $i = \nabla_\lambda E(\lambda, \theta)$.

Winding connection

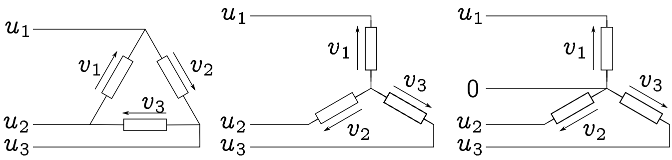


- often, winding voltages v not controlled directly
- (*e.g.*, wye/delta windings, windings contained in rotor)
- indirect control through terminal voltages $u(t) \in \mathbf{R}^m$

$$Ci(t) = 0, \qquad v(t) = C^T e(t) + Bu(t),$$

- $C \in \mathbf{R}^{p \times n}$ is the *connection topology matrix*
- $B \in \mathbf{R}^{n \times m}$ is the *voltage input matrix*
- $e(t) \in \mathbf{R}^p$ are floating node voltages

Winding connection examples



$$Ci(t) = 0, \qquad v(t) = C^T e(t) + Bu(t),$$

- simple delta, wye, and independent winding connections
- some windings may be controlled only through induction
  - *e.g.*, windings on the rotor

Optimal waveform design

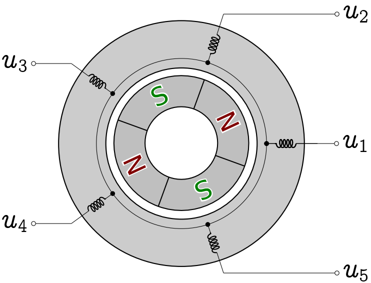
- waveform design problem:

$$\begin{array}{ll} \text{minimize} & p_{\text{loss}} + \gamma r \\ \text{subject to} & \bar{\tau} = \tau_{\text{des}}, \\ & \text{torque equation} \\ & \text{inductance equation} \\ & \text{circuit dynamics} \\ & \text{winding pattern} \end{array}$$

- variables are i, v, u, e, λ, τ (all functions on $\mathbf{R}_+$)
- problem data:
  - tradeoff parameter $\gamma \geq 0$
  - resistance matrix $R \in \mathbf{S}^n_{++}$
  - energy function $E : \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}_+$
  - shaft speed $\omega \in \mathbf{R}$
  - desired torque $\tau_{\text{des}} \in \mathbf{R}$
  - winding connection parameters $B \in \mathbf{R}^{n \times m}$ and $C \in \mathbf{R}^{p \times n}$

- nonconvex in general, due to nonlinear torque and inductance equations
- problem data 2π-periodic, but periodicity of solution not known
  - in practice, solutions often *not* 2π-periodic in θ

Permanent magnet motor



- magnets in rotor change magnetic flux through windings as they pass, producing voltage across the windings
- by simultaneously pushing current through the windings, electrical energy is extracted (or injected)

Permanent magnet motor

- energy function is quadratic:

$$E(\lambda, \theta) = \lambda^T A \lambda + b(\theta)^T \lambda$$

  (quadratic part independent of rotor angle)

- inductance equation is linear:

$$\lambda = Li + \lambda_{\mathrm{mag}}(\theta)$$

  L is the *inductance matrix*, λ_{mag} is the flux due to rotor magnets

- torque equation is affine:

$$\tau = k(\theta)^T i + \tau_{\mathrm{cog}}(\theta)$$

  $k(\theta)$ is the *motor constant*, τ_{cog} is the *cogging torque*

Permanent magnet motor

- dynamics, with λ, are

$$v(t) = Ri(t) + \dot{\lambda}(t)$$

- eliminating λ:

$$v(t) = Ri(t) + L\frac{di}{dt}(t) + \omega k(\theta)$$

Permanent magnet motor, waveform design

- optimal waveform design problem is convex
- 2π-periodicity of problem data with convexity implies 2π-periodicity of a solution, if one exists[3]

[3] Boyd, Vandenberghe. *Convex Optimization*, page 189. 2004

Permanent magnet motor, waveform design

- waveform design problem:

$$
\begin{array}{ll}
\text{minimize} & \overbrace{\frac{1}{2\pi}\int_0^{2\pi} i(\theta)^T R i(\theta)\, d\theta}^{\text{power loss}} + \gamma \overbrace{\frac{1}{2\pi}\int_0^{2\pi} \left(\tau(\theta) - \tau_{\mathrm{des}})^2\right)\, d\theta}^{\text{torque ripple}} \\
\text{subject to} & \frac{1}{2\pi}\int_0^{T} \tau(\theta)\, d\theta = \tau_{\mathrm{des}} \qquad \text{(av. torque)} \\
& \tau = k(\theta)^T i + \tau_{\mathrm{cog}}(\theta) \qquad \text{(torque)} \\
& v(\theta) = Ri(\theta) + \omega L i'(\theta) + \omega k(\theta) \qquad \text{(dynamics)} \\
& Ci(\theta) = 0 \\
& v(\theta) = C^T e(\theta) + Bu(\theta) \qquad \text{(winding conn.)}
\end{array}
$$

- variables are i, v, u, e, τ (all functions on $[0, 2\pi]$)

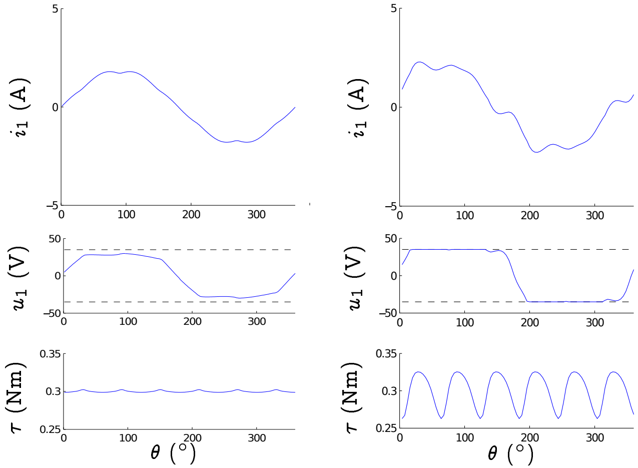
Permanent magnet motor, waveform design

- a periodic linear-quadratic control problem
  - can discretize, solve by least squares
- in fact, many extensions retain convexity:
  - voltage limits $|u(\theta)| \leq u_{\max}$
  - current limits $|i(\theta)| \leq i_{\max}$
  - nonquadratic definitions of torque ripple
- extensions typically involve solving a quadratic program
- more discussion in paper[4]:
  - extensions/variations
  - custom fast solver $\rightarrow$ online waveform generation

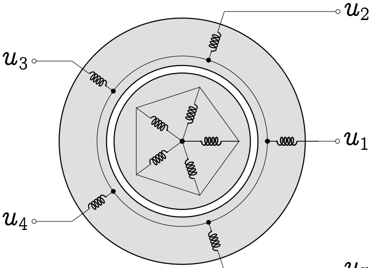
[4] Moehle, Boyd. *Optimal Current Waveforms for Brushless Permanent Magnet Motors.* 2015.

Example

- $\gamma = 2\ \mathrm{W/Nm^2}$
- left: $\omega = 300$ rad/s, right: $\omega = 400$ rad/s



Induction motor



- rotor magnets replaced by more windings, which act as electromagnets (with current)
- rotor current produced my magnetic induction (using stator currents)

Induction motor

- Energy function is again quadratic:

$$E(\lambda, \theta) = \lambda^T A(\theta)\lambda$$

  quadratic part dependent on θ (affine part omitted for simplicity)
- inductance equation is linear:

$$\lambda = L(\theta)i$$

- torque is (indefinite) quadratic:

$$\tau = -i^T L'(\theta)i$$

Induction motor, maximum torque problem

- general waveform design problem intractable
- we focus on the maximum torque problem ($\gamma = 0$):
  - torque ripple penalty disappears
  - maximize average torque (a nonconvex quadratic function)
  - power loss constraint (a convex quadratic function)

Induction motor, maximum torque problem

- waveform design problem:

$$
\begin{array}{lll}
\text{maximize} & \overbrace{\lim_{T\to\infty} \frac{1}{T}\int_0^T -i(t)^T L'(\omega t) i(t)\; dt}^{\text{average torque}} & \\
\text{subject to} & \lim_{T\to\infty} \frac{1}{T}\int_0^T i(t)^T R i(t)\; dt \leq p_{\text{loss}} & \text{(power loss)} \\
& v(t) = R i(t) + \dot{\lambda}(t) & \text{(dynamics)} \\
& C i(t) = 0 & \\
& v(t) = C^T e(t) + B u(t) & \text{(winding conn.)} \\
& \lambda(t) = L(\omega t) i(t) & \text{(induction)}
\end{array}
$$

- variables are i, v, u, e, λ (all functions on $\mathbf{R}_+$)
- equivalent to minimizing p_{loss} with average torque constraint

Induction motor, maximum torque problem

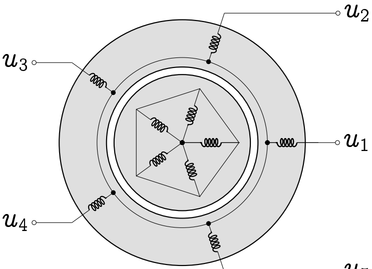
- can be converted to a nonconvex linear-quadratic control problem with a quadratic constraint
  - strong duality holds
  - original proof due to Yakubovich[5]
- further details in our paper[6]
  - equivalent semidefinite program (SDP)
  - method for constructing optimal waveforms from SDP solution
  - proof of tightness

[5] Yakubovich. *Nonconvex optimization problem: The infinite-horizon linear-quadratic control problem with quadratic constraints*. 1992.

[6] Moehle, Boyd. *Maximum Torque-per-Current Control of Induction Motors via Semidefinite Programming*. 2016.

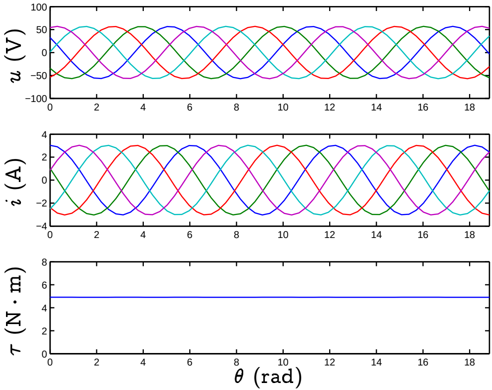
Example

traditional, sinusoidally wound, 5-phase motor with wye winding:



desired torque $\tau_{\mathrm{des}} = 5\ \mathrm{Nm}$, speed $\omega = 50\ \mathrm{rad/s}$

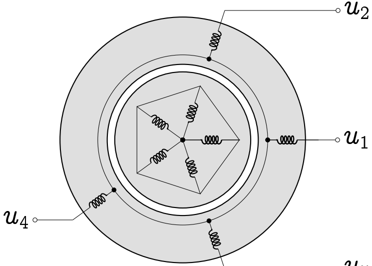
Example



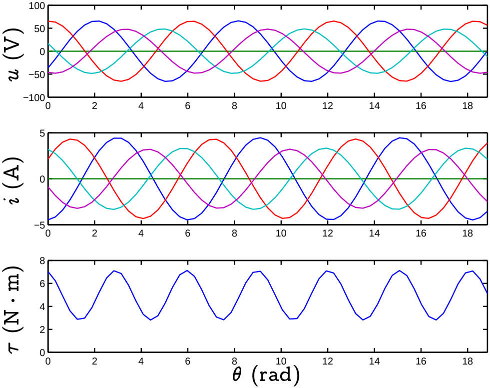
power loss is 11 W per Nm torque produced

Stator fault

Same motor, with open-phase fault:



Stator fault



power loss is 14 W per Nm torque produced

Outline

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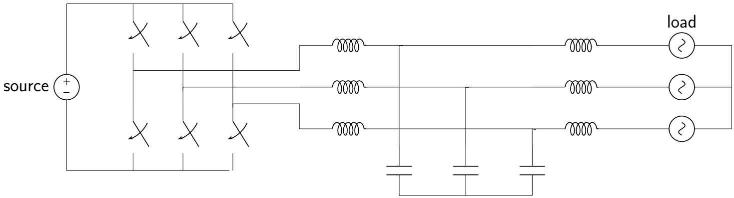
   - permanent magnet
   - induction
2. control of switched-mode converters

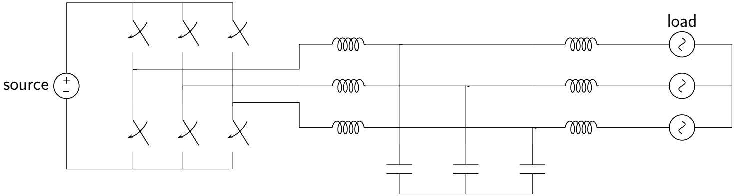
Controlling switched-mode converters



- input are switch configurations
- traditionally:[7]
  1. make discrete input continuous, by considering averaged switch on-time (‘duty cycle’)
  2. choose a duty cycle corresponding to desired equilibrium
  3. linearize the resulting system around equilibrium, use linear control
- now:
  - direct (switch-level) control

[7] Kassakian. *Principles of power electronics*. 1991.

Switched-linear circuit



- state $x_t \in \mathbf{R}^n$ contains inductor currents, capacitor voltages
  - can be augmented to contain, *e.g.*, reference signal
- for each switch configuration, we have a linear circuit
- switched-affine dynamics:

$$x_{t+1} = A^{u_t} x_t + b^{u_t}, \quad t = 0, 1, \ldots,$$

- dynamics specified by A^i, b^i in mode i
- control input is the mode $u_t \in \{1, \ldots, K\}$
- may include mode restrictions (*e.g.*, for a diode)

Switched-affine control

- switched-affine control problem is

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T} g(x_t) \\ \text{subject to} & x_{t+1} = A^{u_t} x_t + b^{u_t} \\ & x_0 = x_{\mathrm{init}} \\ & u_t \in \{1, \ldots, K\} \end{array}$$

- constraints hold for all t
- variables are u_t and $x_t \in \mathbf{R}^n$
- problem data are dynamics A^i, b^i, function g, and initial condition x_{init}
- can be solved by trying out K^T trajectories

'Solution' via dynamic programming

- Bellman recursion: find functions V_t such that

$$V_t(x) = \min_{u \in \{1,\ldots,K\}} g(x) + V_{t+1}(A^u x + b^u)$$

  for all x, for $t = T-1, \ldots, 0$
- final value function $V_T = g$
- optimal problem value is $V_0(x_{\mathrm{init}})$ at initial state x_{init}
- in general, intractable to compute (or store) V_t

Model predictive control

- idea: solve switched-affine control problem, implement first control action u_0, measure new system state, and repeat
- called *model predictive control* (MPC) or *receding horizon control*
- given $V = V_1$, MPC policy satisfies

$$\phi_{\mathrm{mpc}}(x) \in \underset{u \in \{1,\ldots,K\}}{\operatorname{argmin}} V(A^u x + b^u)$$

(ties broken arbitrarily)

Approximate dynamic programming policy

- in practice, MPC policy only works for T small
- (system response time measured in μs)
- instead, approximate V as a quadratic function $\hat{V}$
- given $\hat{V}$, ADP policy satisfies

$$\phi_{\rm adp}(x) \in \mathop{\rm argmin}_{u \in \{1,\ldots,K\}} \hat{V}(A^u x + b^u)$$

- evaluating $\phi_{\rm adp}$ requires evaluating a few quadratic functions

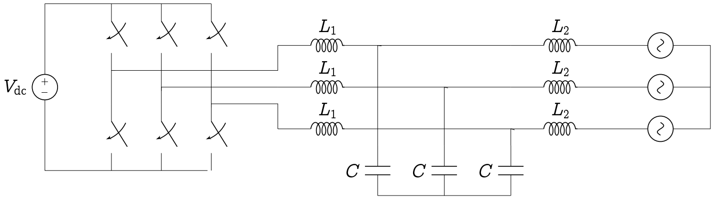
How to obtain $\hat{V}$?

- quadratic lower bounds on V can be found via semidefinite programming[8]
- compute $V(x^{(i)})$ for many states $x^{(i)}$, fit best quadratic function $\hat{V}$
  - we used this method
  - subproblems solved using methods described in paper[9]
- use exact value function for approximate linear control problem (*e.g.*, linear-quadratic control)
  - provides a link to traditional methods

[8]Wang, O'Donoghue, Boyd. *Approximate Dynamic Programming via Iterated Bellman Inequalities*. 2014.

[9]Moehle, Boyd. *A Perspective-Based Convex Relaxation for Switched-Affine Optimal Control*. 2015.

Inverter example



- state x_t are inductor currents and capacitor voltages, and desired output current phasors
- cost function is deviation of output currents from desired (sinusoidally-varying) values
- model parameters $V_{\mathrm{dc}} = 700$ V, $L_1 = 6.5$ μH, $L_2 = 1.5$ μH, $C = 15$ μF, $V_{\mathrm{load}} = 300$ V, and desired output current amplitude $I_{\mathrm{des}} = 10$ A.
- sampling time 30 μs

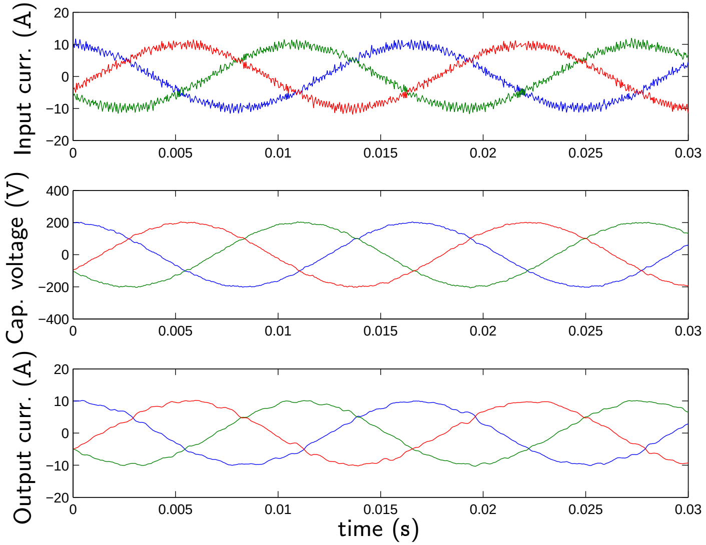
Result

Policy	State cost
ADP policy,	0.70
MPC policy, $T = 1$	∞
MPC policy, $T = 2$	∞
MPC policy, $T = 3$	∞
MPC policy, $T = 4$	∞
MPC policy, $T = 5$	0.45

- for $T < 5$ MPC policy is unstable
- running MPC with $T = 5$ takes several seconds on PC
- ADP takes few hundred flops (can be carried out in μs)

Result

In steady state:



Conclusions

- unconventional motors (asymmetrical, nonsinusoidally-wound, non-rotary) can be controlled using optimization, by designing the waveform to the motor
- modern techniques can be used to generate optimal controllers for power electronic converters, which
  - have fast response
  - can easily incorporate constraints
  - are intuitive to understand and tune
  - make good use of modern microprocessor capabilities

Sources

- motors
  - N. Moehle, S. Boyd. *Optimal Current Waveforms for Brushless Permanent Magnet Motors.* International Journal of Control, 2015.
  - N. Moehle, S. Boyd. *Maximum Torque-per-Current Control of Induction Motors via Semidefinite Programming.* Conf. on Decision and Control, 2016.
  - N. Moehle, S. Boyd. *Optimal Current Waveforms for Switched-Reluctance Motors.* Multi-Conf. on Systems and Control, 2016.
- converters
  - N. Moehle, S. Boyd. *A Perspective-Based Convex Relaxation for Switched-Affine Optimal Control.* Systems and Control Letters, 2015.
  - N. Moehle, S. Boyd. *Value Function Approximation for Direct Control of Switched Power Converters.* Conf. on Industrial Electronics and Applications, 2017.

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